A Unified Review of Optimization

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***Abstract*—The main objective of this paper is to give a survey of recent automatic optimization methods which either have found or should find useful application in the area of computer-aided network design. Huang's family of algorithms for unconstrained optimization is reviewed. The Fletcher method and the Charalambous family of algorithms for unconstrained optimization, which abandon the "full linear search," are presented. Special emphasis is devoted to algorithms by Bandler and Charalambous on least pth and minimax optimization which can be readily programmed and used. Due to work by Bandler and Charalambous, it is shown how constrained minimax problems can be solved exactly as unconstrained minimax problems by using a new approach to nonlinear programming. The application of minimax optimization on the design of lumped-distributed active filters, problems for future investigation, and a select list of references are also included.**

I. Introduction

OPTIMIZATION techniques are of great interest to engineers and applied mathematicians. The former group has a practical or semipractical problem demanding solution, while the latter is challenged primarily by the difficult task of obtaining theoretical conditons for optimality. Optimization techniques are needed in the frequent case when the synthesis procedures of classical theory are for some reason inapplicable, e.g., if the circuit structure to be designed is too complicated to permit a formal synthesis procedure. Over the past decade, there has been a steady shift in applied optimization from the status of an art to that of a scientific discipline. To a large degree this shift is due to the development of high-speed computers and of fast optimization algorithms. This paper presents some recent automatic optimization methods which have found or should find useful application in the area of computer-aided network design.

It will be apparent from the unified treatment of gradient algorithms for unconstrained optimization due to Huang [1] and to a recent theorem by Dixon [2], why there has not been much improvement in the area of unconstrained optimization from 1963 to 1970. Also, from Property 1 of Fletcher [3], it will be clear why some workers have reported success with the Fletcher–Powell algorithm [4] without "full linear search." The Charalambous family of algorithms [5] for unconstrained optimization, which is based on homogeneous models, is also reviewed.

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Section III presents a unified review of least pth and minimax optimization due to Bandler and Charalambous. The difficulty of using least pth approximation in cases when we have upper and lower response specification has been completely eliminated by using the Bandler–Charalambous generalized least pth objective function [6]. Furthermore, by using a simple scaling procedure, it is possible to overcome the ill-conditioning of the objective function for very large values of p and still have reasonably well-conditioned objective functions [7]. Large values of p are required so that the least pth optimal solution is very close to the optimal minimax solution [8]–[11]. Using the generalized least pth objective function, the necessary and sufficient conditions for minimax optimization can be derived [12]–[16].

Unlike the usual case in which the value of p has to tend to infinity so as to be able to get results very close to a minimax solution, Charalambous and Bandler very recently proposed two new algorithms for minimax optimization in which *any* value of p greater than one can be used to obtain the minimax optimum [17], [18]. Also it will be shown that if we are investigating whether a particular structure will satisfy design specifications in the minimax sense, any single suitable least pth optimization will reveal this!

From the results of Section IV it will be clear how any suitable algorithm for unconstrained optimization, nonlinear unconstrained minimax optimization, least pth optimization, or nonlinear programming can be used to solve both the minimax optimization with constraints and the nonlinear programming problem [17], [19]–[21].

This paper is intended to be an extension of the review paper presented by Bandler for the 1969 Special Issue of the IEEE Transactions on Microwave Theory and Techniques on computer-oriented microwave practices [9], where he thoroughly covers one-dimensional optimization methods and multidimensional direct-search optimization methods. For this reason these methods are not going to be considered in this paper (see also [22]). Most of the material presented in this paper is based on the author's Ph.D. work [23].

II. Unconstrained Optimization

A. Fundamental Concepts and Definitions

The unconstrained optimization problem is to calculate the minimum value of the scalar valued function U where

$$U \triangleq U(\boldsymbol{\phi}) \tag{1}$$

and

$$\boldsymbol{\phi} \triangleq [\phi_1 \phi_2 \cdots \phi_k]^T. \quad (2)$$

U is called the *objective function*, and the column vector $\boldsymbol{\phi}$ contains the k real independent variables. The term "unconstrained" implies that the value of each variable can be any real number. Maximizing a function is the same as minimizing the negative of the function, so only the minimization problem will be considered.

A point $\check{\boldsymbol{\phi}}$ is called a *global* minimum of $U(\boldsymbol{\phi})$ if

$$U(\check{\boldsymbol{\phi}}) \leq U(\boldsymbol{\phi}) \quad (3)$$

for all $\boldsymbol{\phi}$. If the strict inequality holds for $\boldsymbol{\phi} \neq \check{\boldsymbol{\phi}}$, the minimum is said to be *unique*. If (3) holds only in the neighborhood of $\check{\boldsymbol{\phi}}$, then $\check{\boldsymbol{\phi}}$ is called a *local* minimum of U.

The first three terms of the multidimensional Taylor series are given by

$$U(\boldsymbol{\phi} + \Delta\boldsymbol{\phi}) = U(\boldsymbol{\phi}) + \nabla^T U(\boldsymbol{\phi}) \Delta\boldsymbol{\phi} + \tfrac{1}{2} \Delta\boldsymbol{\phi}^T \boldsymbol{G}(\boldsymbol{\phi}) \Delta\boldsymbol{\phi} + \cdots \quad (4)$$

where

$$\Delta\boldsymbol{\phi} \triangleq [\Delta\phi_1 \Delta\phi_2 \cdots \Delta\phi_k]^T \quad (5)$$

represents the incremental change in the parameters,

$$\nabla \triangleq \left[\frac{\partial}{\partial \phi_1} \frac{\partial}{\partial \phi_2} \cdots \frac{\partial}{\partial \phi_k}\right]^T \quad (6)$$

is the first partial derivative operator with respect to the parameter vector $\boldsymbol{\phi}$, and

$$\boldsymbol{G} \triangleq \nabla(\nabla^T U) \quad (7)$$

is the matrix of second partial derivatives, the Hessian matrix, which is symmetric if it exists.

Assuming the first and second partial derivatives exist, a point $\check{\boldsymbol{\phi}}$ is a minimum of U if

$$\nabla U(\check{\boldsymbol{\phi}}) = 0 \quad (8)$$

and the Hessian matrix is positive semidefinite at the point $\check{\boldsymbol{\phi}}$. This can be seen from (4).

Considering the first three terms of the Taylor series expansion about the point $\check{\boldsymbol{\phi}}$, and bearing in mind (8), we have

$$U(\boldsymbol{\phi}) \doteq \tfrac{1}{2}(\boldsymbol{\phi} - \check{\boldsymbol{\phi}})^T \boldsymbol{G}(\check{\boldsymbol{\phi}})(\boldsymbol{\phi} - \check{\boldsymbol{\phi}}) + U(\check{\boldsymbol{\phi}}). \quad (9)$$

Thus the function behaves like a pure quadratic in the vicinity of $\check{\boldsymbol{\phi}}$.

B. Multidimensional Gradient Strategies

In the rest of this section, methods are described which utilize only the information of the first partial derivatives to determine the direction of search to the minimum of a differentiable function.

The iterative scheme, in general, is to find

$$\{\boldsymbol{\phi}^0, \boldsymbol{\phi}^1, \cdots, \boldsymbol{\phi}^k, \cdots\}$$

such that

$$\boldsymbol{\phi}^{i+1} = \boldsymbol{\phi}^i + \boldsymbol{\delta}^i \quad (10)$$

and

$$\lim_{i \to \infty} \boldsymbol{g}(\boldsymbol{\phi}^i) = \boldsymbol{0} \quad (11)$$

where

$$\boldsymbol{\delta}^i = \alpha_i \boldsymbol{d}^i \quad (12)$$

$$\boldsymbol{g} = \nabla U. \quad (13)$$

$\boldsymbol{d}^i$ is a k-dimensional vector which denotes the direction of search and α_i a scalar which takes the value of λ minimizing $U(\boldsymbol{\phi}^i + \lambda \boldsymbol{d}^i)$ along the direction $\boldsymbol{d}^i$, resulting in

$$(\boldsymbol{g}^{i+1})^T \boldsymbol{d}^i = 0. \quad (14)$$

Considering the first two terms of the Taylor series expansion about the point $\boldsymbol{\phi}^i$, we have

$$U(\boldsymbol{\phi}^i + \lambda \boldsymbol{d}^i) = U(\boldsymbol{\phi}^i) + \lambda (\boldsymbol{g}^i)^T \boldsymbol{d}^i. \quad (15)$$

By definition, the direction $\lambda \boldsymbol{d}^i$ is a "downhill direction" if

$$\lambda(-\boldsymbol{g}^i)^T \boldsymbol{d}^i > 0. \quad (16)$$

In other words, the sign of α_i is the same as that of $(-\boldsymbol{g}^i)^T \boldsymbol{d}^i$.

C. Huang's Generalized Algorithm [1]

If $\boldsymbol{d}^i = -(\boldsymbol{G}^i)^{-1} \boldsymbol{g}^i$, we have the Newton algorithm, and if $\boldsymbol{d}^i = -\boldsymbol{g}^i$, we have the steepest descent algorithm. Newton's method has an excellent rate of convergence, if it converges, but the method may not converge at all, and it requires the second derivatives of the function to be minimized. On the other hand, the steepest descent method is superior to Newton's method in stability and requires only the first derivatives of the function, but convergence is often very slow. Methods which combine the good characteristics of these two methods and use only first derivatives have been developed and are still being developed. Common features of these algorithms include the iterative approximation of the Hessian matrix and the use of conjugacy properties to determine directions of search.

In 1970 Huang derived a general algorithm which is based on the two ideas mentioned previously, and has the property that it will terminate in at most k iterations on quadratic functions. Most of the existing conjugate-gradient algorithms and variable-metric algorithms can be obtained as particular cases.

Huang's generalized algorithm is based on the quadratic model

$$U(\boldsymbol{\phi}) = \tfrac{1}{2}(\boldsymbol{\phi} - \check{\boldsymbol{\phi}})^T \boldsymbol{G}(\boldsymbol{\phi} - \check{\boldsymbol{\phi}}) + U(\check{\boldsymbol{\phi}}). \quad (17)$$

The reason for a quadratic model is that it is the simplest differentiable function that can have a well-defined minimum.

From (17)

$$\boldsymbol{g}(\boldsymbol{\phi}) = \boldsymbol{G}(\boldsymbol{\phi} - \check{\boldsymbol{\phi}})$$

$$\boldsymbol{\gamma}^i = \boldsymbol{G}\boldsymbol{\delta}^i = \alpha_i \boldsymbol{G}\boldsymbol{d}^i \quad (18)$$

where

$$\boldsymbol{\gamma}^i \triangleq \boldsymbol{g}^{i+1} - \boldsymbol{g}^i. \quad (19)$$

From (18) and (14), we get

$$\alpha_i = \frac{(-g^i)^T d^i}{(d^i)^T G d^i} \tag{20}$$

and from (18) and (20),

$$(g^l)^T d^j = \sum_{i=j+1}^{l-1} \alpha_i (d^i)^T G d^j, \qquad j = 0,1,\cdots,l-2. \tag{21}$$

A set of nonzero vectors $\{d^0,d^1,\cdots,d^{k-1}\}$ are G *conjugate* if

$$(d^i)^T G d^j = 0, \qquad i \neq j, i,j = 0,1,\cdots,k-1. \tag{22}$$

If G is positive definite, the k vectors $\{d^0,d^1,\cdots,d^{k-1}\}$, which are G conjugate, are also *linearly independent*. Therefore, from (14), (21), and (22),

$$(g^l)^T d^j = 0, \qquad j = 0,1,\cdots,l-1. \tag{23}$$

Let $l = k$, then

$$(g^k)^T d^j = 0, \qquad j = 0,1,\cdots,k-1. \tag{24}$$

Since the elements of $\{d^0,d^1,\cdots,d^{k-1}\}$ are linearly independent,

$$g^k = \mathbf{0}. \tag{25}$$

Therefore the minimum of a positive definite quadratic function is attained in at most k iterations.

Let

$$d^i = -(H^i)^T g^i \tag{26}$$

where H^i is a $k \times k$ matrix. From (22),

$$(g^i)^T H^i \gamma^j = 0, \qquad j = 0,1,\cdots,i-1 \tag{27}$$

and, from (23),

$$(g^i)^T \delta^j = 0, \qquad j = 0,1,\cdots,i-1. \tag{28}$$

From (27) and (28),

$$H^i \gamma^j = \rho \delta^j, \qquad j = 0,1,\cdots,i-1 \tag{29}$$

where ρ is an arbitrary constant.

Let

$$H^{i+1} = H^i + \Delta H^i \tag{30}$$

then, from (29) and (30),

$$\Delta H^i \gamma^j = \mathbf{0}, \qquad j = 0,1,\cdots,i-1 \tag{31}$$

$$\Delta H^i \gamma^i = \rho \delta^i - H^i \gamma^i. \tag{32}$$

To satisfy (32), one can choose

$$\Delta H^i = \rho \frac{\delta^i (c_1^i)^T}{(c_1^i)^T \gamma^i} - \frac{H^i \gamma^i (c_2^i)^T}{(c_2^i)^T \gamma^i} \tag{33}$$

where c_1^i and c_2^i are arbitrary k-dimensional column vectors satisfying $(c_1^i)^T\gamma^i \neq 0$ and $(c_2^i)^T\gamma^i \neq 0$, respectively. Also, ΔH^i satisfies condition (31) if

$$(c_1^i)^T \gamma^j = 0 \quad \text{and} \quad (c_2^i)^T \gamma^j = 0, \qquad j = 0,1,\cdots,i-1. \tag{34}$$

Also, from (12), (18), (22), and (29), we have

$$(\delta^i)^T \gamma^j = 0 \quad \text{and} \quad (\gamma^i)^T H^i \gamma^j = 0, \qquad j = 0,1,\cdots,i-1. \tag{35}$$

Therefore,

$$\begin{aligned} c_1^i &= C_1^i \delta^i + C_2^i (H^i)^T \gamma^i \\ c_2^i &= K_1^i \delta^i + K_2^i (H^i)^T \gamma^i \end{aligned} \tag{36}$$

where C_1^i, C_2^i, K_1^i, and K_2^i are scalars. Substituting for c_1^i and c_2^i in (33), we obtain Huang's generalized algorithm

$$\left[H^{i+1} = H^i + \rho \frac{\delta^i (C_1^i \delta^i + C_2^i (H^i)^T \gamma^i)^T}{(C_1^i \delta^i + C_2^i (H^i)^T \gamma^i)^T \gamma^i} - \frac{H^i \gamma^i (K_1^i \delta^i + K_2^i (H^i)^T \gamma^i)^T}{(K_1^i \delta^i + K_2^i (H^i)^T \gamma^i)^T \gamma^i} \right]. \tag{37}$$

When $i = k$,

$$H^k G d^j = \rho d^j, \qquad j = 0,1,\cdots,k-1. \tag{38}$$

Since the elements of $\{d^0,d^1,\cdots,d^{k-1}\}$ are linearly independent,

$$H^k = \rho G^{-1}. \tag{39}$$

Therefore, H^k is a symmetric matrix. If ρ is positive, H^k is positive definite. If ρ is zero, the matrix H^k is the null matrix, and if ρ is negative, the matrix H^k is negative definite.

Special Case 1: If we let $\rho = 1$, $C_1^i = 1$, $C_2^i = 0$, $K_1^i = 0$, and $K_2^i = 1$, then

$$H^{i+1} = H^i + \frac{\delta^i (\delta^i)^T}{(\delta^i)^T \gamma^i} - \frac{H^i \gamma^i (\gamma^i)^T H^i}{(\gamma^i)^T H^i \gamma^i}. \tag{40}$$

This is the Fletcher and Powell [4] updating formula. This updating formula has the property that if H^0 is a positive definite symmetric matrix, then H^i is also a symmetric positive definite matrix, i.e., $\alpha_i > 0$.

Special Case 2: If we let

$$\rho = 1, \frac{C_2^i}{C_1^i} = \frac{-(\delta^i)^T \gamma^i}{(\delta^i)^T \gamma^i + (\gamma^i)^T H^i \gamma^i}, K_1^i = 1, \text{ and } K_2^i = 0$$

then

$$H^{i+1} = H^i - \frac{\delta^i (\gamma^i)^T H^i}{(\delta^i)^T \gamma^i} - \frac{H^i \gamma^i (\delta^i)^T}{(\delta^i)^T \gamma^i} + \left(1 + \frac{(\gamma^i)^T H^i \gamma^i}{(\delta^i)^T \gamma^i}\right) \frac{\delta^i (\delta^i)^T}{(\delta^i)^T \gamma^i}. \tag{41}$$

This updating formula was discovered by Fletcher [3], Broyden [24], and Goldfarb [25] and has the same properties as that of Fletcher and Powell.

Dixon [2] proved the following results for a general nonlinear function:

$$(H^i)^T g^i = \beta_i q^i \tag{42}$$

where β_i is a scalar defined by

$$\beta_i = -(\alpha_{i-1}K_1^{i-1} - K_2^{i-1}) \cdot \frac{(\boldsymbol{\gamma}^{i-1})^T(\boldsymbol{H}^{i-1})^T\boldsymbol{g}^{i-1}}{(K_1^{i-1}\boldsymbol{\delta}^{i-1} + K_2^{i-1}(\boldsymbol{H}^{i-1})^T\boldsymbol{\gamma}^{i-1})^T\boldsymbol{\gamma}^{i-1}} \tag{43}$$

and $\boldsymbol{q}^i$ is a k-dimensional vector defined by

$$\boldsymbol{q}^i = \left(\prod_{j=0}^{i-1}\left(\boldsymbol{I} - \frac{\boldsymbol{\delta}^j(\boldsymbol{\gamma}^j)^T}{(\boldsymbol{\delta}^j)^T\boldsymbol{\gamma}^j}\right)\right)(\boldsymbol{H}^0)^T\boldsymbol{g}^i + \sum_{j=0}^{i-2}\left(\prod_{l=j+1}^{i-1}\left(\boldsymbol{I} - \frac{\boldsymbol{\delta}^l(\boldsymbol{\gamma}^l)^T}{(\boldsymbol{\delta}^l)^T\boldsymbol{\gamma}^l}\right)\right)\frac{\rho(\boldsymbol{\delta}^j)^T\boldsymbol{g}^i}{(\boldsymbol{\delta}^j)^T\boldsymbol{\gamma}^j}\boldsymbol{\delta}^j. \tag{44}$$

From (26) and (42), it can be seen that

$$\boldsymbol{d}^i = -\beta_i\boldsymbol{q}^i. \tag{45}$$

Equations (44) and (45) show that $\boldsymbol{d}^i$ depends only on the initial matrix $\boldsymbol{H}^0$ and the value of ρ, and is independent of the parameters C_1^i, C_2^i, K_1^i, and K_2^i. Therefore, if a sequence of points $\boldsymbol{\phi}^i$ is generated using a group of formulas belonging to Huang's family on the same general nonquadratic function, then the necessary and sufficient conditions for all the sequences to be identical is that all formulas in the group possess the same value of ρ and the same value of the matrix $\boldsymbol{H}^0$.

Since most of the algorithms introduced by most of the authors are members of Huang's family with $\rho = 1$, they should all give identical sequences of points if "full linear search" is used. Due to the fact that the minimum in a one-dimensional search cannot be found exactly, the sequence of points generated might not be exactly the same. Bearing all of the above in mind, it is clear why there was little improvement in the area of unconstrained optimization from 1963, when the Fletcher–Powell algorithm was published, until 1970, when Fletcher came up with the brilliant idea of avoiding the "full linear search" subproblem.

D. Fletcher Algorithm [3]

The main disadvantage of Huang's generalized algorithm is the need to solve the subproblem of finding α_i at each iteration (the full linear search). The importance of the full linear search is that it furnishes a property which enables finite termination to be proved for quadratic functions.

Fletcher replaced the property of quadratic termination by a property which requires that, for a quadratic function with $\boldsymbol{G}$ strictly positive definite, the eigenvalues of $\boldsymbol{H}$ must tend monotonically to those of $\boldsymbol{G}^{-1}$ in certain sense (he calls this Property 1). He has further shown that (40) and (41) and any convex combination of them possess Property 1. Because the Fletcher and Powell updating formula satisfies Property 1 of Fletcher, it is clear why workers had success in using this algorithm without full linear search.

Use of (40) alone might cause $\boldsymbol{H}$ to become singular. For this reason a choice is made between the two updating formulas by the following test: if

$$(\boldsymbol{\delta}^i)^T\boldsymbol{\gamma}^i < (\boldsymbol{\gamma}^i)^T\boldsymbol{H}^i\boldsymbol{\gamma}^i \tag{46}$$

then formula (40) is used; otherwise, formula (41) is used. In the Fletcher method, the correction $\boldsymbol{\delta}$ is determined by $\boldsymbol{\delta} = -\lambda\boldsymbol{H}\boldsymbol{g}$, where the value of λ is chosen to satisfy the following inequality:

$$\mu \le \frac{\Delta U}{\boldsymbol{g}^T\boldsymbol{\delta}} \le 1 - \mu \tag{47}$$

where μ is a preassigned small quantity. Note that the expression between the inequality signs tends to one as λ tends to zero, and that if $U(\boldsymbol{\phi})$ is bounded below, this expression tends to zero or to become negative as λ tends to infinity. Therefore, a suitable value of λ exists. Fletcher is confident that the choice of $\lambda = 1$ is not too small. Therefore, he tries the estimates $\lambda = 10^{-j}$ for $j = 0,1,\cdots$, and he accepts the first estimate that satisfies the left-hand inequality of expression (47). In practice, he finds that on most iterations the value of $\lambda = 1$ is satisfactory.

A simple interpretation of which formula should be used is given in the following equation:

$$\boldsymbol{H}_f = \boldsymbol{H}_{fp} + \boldsymbol{v}\boldsymbol{v}^T \tag{48}$$

where

$$\boldsymbol{v} = (\boldsymbol{\gamma}^T\boldsymbol{H}\boldsymbol{\gamma})^{1/2}\left(\frac{\boldsymbol{\delta}}{\boldsymbol{\delta}^T\boldsymbol{\gamma}} - \frac{\boldsymbol{H}\boldsymbol{\gamma}}{\boldsymbol{\gamma}^T\boldsymbol{H}\boldsymbol{\gamma}}\right) \tag{49}$$

and $\boldsymbol{H}_f$, $\boldsymbol{H}_{fp}$ denote the Fletcher, and Fletcher and Powell updating formulas, respectively. Let us suppose that the function is quadratic; then replacing $\boldsymbol{\delta}$ by $\boldsymbol{G}^{-1}\boldsymbol{\gamma}$, the inequality (46) becomes $\boldsymbol{\gamma}^T\boldsymbol{G}^{-1}\boldsymbol{\gamma} < \boldsymbol{\gamma}^T\boldsymbol{H}_\gamma$. This shows that $\boldsymbol{H}$ is "larger" than $\boldsymbol{G}^{-1}$, and therefore the "smaller" formula $\boldsymbol{H}_{fp}$ is used. If $\boldsymbol{\gamma}^T\boldsymbol{G}^{-1}\boldsymbol{\gamma} > \boldsymbol{\gamma}^T\boldsymbol{H}_\gamma$, then $\boldsymbol{H}$ is "smaller" than $\boldsymbol{G}^{-1}$, and therefore the "larger" formula $\boldsymbol{H}_f$ is used. If equality holds, then no indication is given which formula to use. In this case, $\boldsymbol{H}_f$ is used to avoid possible singularity in $\boldsymbol{H}$.

E. Charalambous Family of Algorithms [5]

Jacobson and Oksman [26] derived an algorithm based on the homogeneous model

$$U(\boldsymbol{\phi}) = \frac{1}{\gamma}(\boldsymbol{\phi} - \check{\boldsymbol{\phi}})^T\boldsymbol{g}(\boldsymbol{\phi}) + U(\check{\boldsymbol{\phi}}) \tag{50}$$

where γ is the degree of homogeneity. Note that if $\gamma = 2$ and the Hessian matrix is constant, we have the quadratic model discussed previously.

Charalambous [5] presented a family of algorithms based on (50). From (50),

$$\boldsymbol{y}^T\boldsymbol{\alpha} = v \tag{51}$$

where

$$v \triangleq \boldsymbol{\phi}^T\boldsymbol{g}(\boldsymbol{\phi})$$

$$\boldsymbol{y}^T \triangleq [\boldsymbol{g}^T(\boldsymbol{\phi})U(\boldsymbol{\phi}) \;\; -1]$$

$$\boldsymbol{\alpha}^T \triangleq [\check{\boldsymbol{\phi}}^T \gamma w]$$

$$w = \gamma U(\check{\boldsymbol{\phi}}).$$

Note that $\boldsymbol{\alpha}$ contains the optimum parameter vector. At some point $\boldsymbol{\phi}^i$,

$$(\boldsymbol{y}^i)^T\boldsymbol{\alpha}^i = v_i. \tag{52}$$

At any step we should satisfy (52) for all steps so far made. Therefore,

$$\boldsymbol{Y}^i\boldsymbol{\alpha}^i = \boldsymbol{v}^i \tag{53}$$

where

$$\boldsymbol{Y}^i \triangleq \begin{bmatrix} (\boldsymbol{y}^1)^T \\ \cdot \\ \cdot \\ \cdot \\ (\boldsymbol{y}^i)^T \end{bmatrix} \qquad \boldsymbol{v}^i \triangleq \begin{bmatrix} v_1 \\ \cdot \\ \cdot \\ \cdot \\ v_i \end{bmatrix}.$$

Using the ideas of Penrose [27], Charalambous derived a very general recursive formula for $\boldsymbol{\alpha}^i$. A special case of the general formula is the following:

$$\boldsymbol{\alpha}^{i+1} = \boldsymbol{\alpha}^i + \frac{\boldsymbol{P}^i\boldsymbol{a}^{i+1}(v_{i+1} - (\boldsymbol{y}^{i+1})^T\boldsymbol{\alpha}^i)}{(\boldsymbol{y}^{i+1})^T\boldsymbol{P}^i\boldsymbol{a}^{i+1}} \tag{54}$$

$$\boldsymbol{P}^{i+1} = \boldsymbol{P}^i - \frac{\boldsymbol{P}^i\boldsymbol{a}^{i+1}(\boldsymbol{y}^{i+1})^T\boldsymbol{P}^i}{(\boldsymbol{y}^{i+1})^T\boldsymbol{P}^i\boldsymbol{a}^{i+1}} + \boldsymbol{A}^{i+1} \tag{55}$$

where $\boldsymbol{A}^j$ is a $(k+2) \times (k+2)$ matrix, and $\boldsymbol{a}^i$ is a $k+2$ vector satisfying

$$\boldsymbol{A}^j\boldsymbol{a}^{n+1} = \boldsymbol{0}, \qquad \begin{matrix} n = 1,\cdots,i \\ j = 1,\cdots,n. \end{matrix} \tag{56}$$

$\boldsymbol{P}_0$ and $\boldsymbol{\alpha}_0$ can have any values, but for simplicity the value $\boldsymbol{P}_0 = \boldsymbol{I}$ is used.

Let $\boldsymbol{A}^i = \boldsymbol{c}^i(\boldsymbol{d}^i)^T$ where $\boldsymbol{c}^i$ and $\boldsymbol{d}^i$ are $k+2$ vectors. It is natural to choose $\boldsymbol{d}^{i+1} = \boldsymbol{a}^{i+1} = \boldsymbol{e}^{i+1}$, where $\boldsymbol{e}^{i+1}$ is a unit vector of the same dimension as $\boldsymbol{a}^{i+1}$ having unity at the $(i+1)$th element. Then (56) is satisfied, and it is independent of $\boldsymbol{c}^i$. Therefore, $\boldsymbol{c}^i$ can have any value. Equations (54) and (55) now become

$$\boldsymbol{\alpha}^{i+1} = \boldsymbol{\alpha}^i + \frac{\boldsymbol{P}^i\boldsymbol{e}^{i+1}(v_{i+1} - (\boldsymbol{y}^{i+1})^T\boldsymbol{\alpha}^i)}{(\boldsymbol{y}^{i+1})^T\boldsymbol{P}^i\boldsymbol{e}^{i+1}} \tag{57}$$

$$\boldsymbol{P}^{i+1} = \boldsymbol{P}^i - \frac{\boldsymbol{P}^i\boldsymbol{e}^{i+1}(\boldsymbol{y}^{i+1})^T\boldsymbol{P}^i}{(\boldsymbol{y}^{i+1})^T\boldsymbol{P}^i\boldsymbol{e}^{i+1}} + \boldsymbol{c}^{i+1}(\boldsymbol{e}^{i+1})^T. \tag{58}$$

Substituting

$$\boldsymbol{c}^{i+1} \text{ by } \lambda \frac{\boldsymbol{P}^i\boldsymbol{e}^{i+1}}{(\boldsymbol{y}^{i+1})^T\boldsymbol{P}^i\boldsymbol{e}^{i+1}}$$

where λ is any scalar quantity, we have

$$\boldsymbol{P}^{i+1} = \boldsymbol{P}^i - \frac{\boldsymbol{P}^i\boldsymbol{e}^{i+1}((\boldsymbol{y}^{i+1})^T\boldsymbol{P}^i - \lambda(\boldsymbol{e}^{i+1})^T)}{(\boldsymbol{y}^{i+1})^T\boldsymbol{P}^i\boldsymbol{e}^{i+1}}. \tag{59}$$

If $\lambda = 1$, then we have the Jacobson–Oksman algorithm [26].

Some important properties of Charalambous family of algorithms are 1) they do not require finding minima along one-dimensional searches; 2) they converge in $k+2$ iterations on homogeneous functions; 3) they do not require the Hessian matrix to be nonsingular; and 4) if $\boldsymbol{\alpha}_1^{i+1}$ and $\boldsymbol{\alpha}_2^{i+1}$ are two updating formulas, then $\mu\boldsymbol{\alpha}_1^{i+1} + (1-\mu)\boldsymbol{\alpha}_2^{i+1}$ will be an updating formula where μ can have any finite value.

F. Termination Criteria

Algorithms terminate after one or more of the following criteria are satisfied: 1) if the change in the objective function becomes less than ϵ_1, a small positive number; 2) if the absolute values of the elements of the increment vector become less than ϵ_2, a small positive number; 3) if the norm of the gradient vector becomes less than ϵ_3, another small positive number. As a safeguard the algorithm should go through k iterations after the terminating criterion is satisfied, before the program terminates. Even if all of the above criteria are satisfied, it will be safer if we make a small perturbation from a point which satisfies the previous criterion and continue the iteration from the perturbed point. If the point ultimately reached from the perturbed point is substantially different from that obtained originally, then it is wise to treat any results with suspicion.

Some other interesting papers on unconstrained optimization are given in [28]–[35].

III. Nonlinear Least pth and Minimax Optimization

Consider a system of m real nonlinear functions

$$f_i(\boldsymbol{\phi}), \qquad i \in I \tag{60}$$

where

$$I \triangleq \{1,2,\cdots,m\}.$$

Let

$$M_f(\boldsymbol{\phi}) \triangleq \max_{i\in I} f_i(\boldsymbol{\phi}). \tag{61}$$

The problem of minimax optimization of (60) consists of finding a point $\check{\boldsymbol{\phi}}$ such that

$$M_f(\check{\boldsymbol{\phi}}) \le M_f(\check{\boldsymbol{\phi}})$$

for all points $\boldsymbol{\phi}$ at least in the neighborhood of $\check{\boldsymbol{\phi}}$.

Various algorithms have been proposed for solving the above problem. Some of the most relevant make use of the generalized least pth objective function of Bandler and Charalambous [6], [7].

A. Bandler–Charalambous Generalized Least pth Objectives [6], [7]

If $f_i(\boldsymbol{\phi}) \ge 0$ for $i \in I$, then it is very well known that

$$M_f(\boldsymbol{\phi}) = \lim_{p\to\infty} U_p^+(\boldsymbol{\phi}) \tag{62}$$

where

$$U_p^+(\boldsymbol{\phi}) = (\sum_{i\in I} (f_i(\boldsymbol{\phi}))^p)^{1/p}. \tag{63}$$

Bandler and Charalambous considered the most general case, in which some of the f_i for $i \in I$ are nonnegative or all of the f_i for $i \in I$ are negative. As we shall see later, this general case occurs in many engineering problems, such as in filter design problems. In the case where at least one of the f_i is nonnegative,

$$M_f(\boldsymbol{\phi}) = \lim_{p\to\infty} U_p^+(\boldsymbol{\phi}) \tag{64}$$

where in this case,

$$U_p^+(\boldsymbol{\phi}) = \Big(\sum_{i \in J(\phi)} (f_i(\boldsymbol{\phi}))^p\Big)^{1/p} \tag{65}$$

and

$$J(\boldsymbol{\phi}) = \{i \mid f_i(\boldsymbol{\phi}) \geq 0, \qquad i \in I\}. \tag{66}$$

In other words, in forming $U_p^+(\boldsymbol{\phi})$, we consider only the functions which are nonnegative.

In the case where $f_i(\boldsymbol{\phi}) < 0$ for all $i \in I$,

$$M_f(\boldsymbol{\phi}) = \lim_{p\to\infty} U_p^-(\boldsymbol{\phi}) \tag{67}$$

where

$$U_p^-(\boldsymbol{\phi}) = -\Big(\sum_{i\in I} (-f_i(\boldsymbol{\phi}))^{-p}\Big)^{-1/p}. \tag{68}$$

In other words, in forming $U_p^-(\boldsymbol{\phi})$, all the functions have to be considered.

This shows that by minimizing the objective functions given by (65) and (68) with very large values of p, we should obtain results very close to the minimax optimum. Without any modification this will apply only in theory, due to ill-conditioning resulting from the numerical evaluation of $[\pm f_i(\boldsymbol{\phi})]^{\pm p}$ for very large values of p. Bandler and Charalambous [7] were able not only to alleviate this ill-conditioning problem, but also to combine the two objective functions given by (65) and (68) into the one objective function, namely,

$$U_p(\boldsymbol{\phi}) = \begin{cases} M_f(\boldsymbol{\phi}) \left(\sum_{i\in K(\phi)} \left(\dfrac{f_i(\boldsymbol{\phi})}{M_f(\boldsymbol{\phi})}\right)^q\right)^{1/q}, & \text{for } M_f(\boldsymbol{\phi}) \neq 0 \\ 0, & \text{for } M_f(\boldsymbol{\phi}) = 0 \end{cases} \tag{69}$$

where

$$q \triangleq \frac{M_f(\boldsymbol{\phi})}{|M_f(\boldsymbol{\phi})|}\cdot p \begin{cases} 1 < p < \infty, & \text{for } M_f > 0 \\ 1 \leq p < \infty, & \text{for } M_f < 0 \end{cases} \tag{70}$$

$$K(\boldsymbol{\phi}) \triangleq \begin{cases} J(\boldsymbol{\phi}), & \text{if } M_f > 0 \\ I, & \text{if } M_f < 0. \end{cases} \tag{71}$$

The gradient vector of the combined objective function is given by

$$\nabla U_p(\boldsymbol{\phi}) = \left(\sum_{i\in K(\phi)} \left(\frac{f_i(\boldsymbol{\phi})}{M_f(\boldsymbol{\phi})}\right)^q\right)^{(1/q)-1} \cdot \sum_{i\in K(\phi)} \left(\frac{f_i(\boldsymbol{\phi})}{M_f(\boldsymbol{\phi})}\right)^{q-1} \nabla f_i(\boldsymbol{\phi}), \qquad \text{for } M_f(\boldsymbol{\phi}) \neq 0. \tag{72}$$

From (69) and (72) it can be seen that if $f_i(\boldsymbol{\phi})$ for $i \in I$ are continuous with continuous first partial derivatives, then, under the stated conditions, the objective function is continuous everywhere with continuous first partial derivatives (except possibly when $M_f(\boldsymbol{\phi}) = 0$, and two or more maxima are equal). Therefore, very efficient gradient optimization algorithms can be used to optimize (69). To overcome the difficulty which arises when $M_f(\boldsymbol{\phi}) = 0$ and two or more maxima are equal (and due to other reasons which will become apparent later), Bandler and Charalambous minimize $M_f'(\boldsymbol{\phi})$, where

$$M_f'(\boldsymbol{\phi}) \triangleq \max_{i\in I} f_i'(\boldsymbol{\phi}) = \max_{i\in I}\,(f_i(\boldsymbol{\phi}) - \xi) \tag{73}$$

where ξ is an artificial margin which is kept constant through the optimization. If the algorithm gets stuck, we increase the value of ξ by a small amount and restart the optimization process. It is important to note that the parameter ξ does not affect the location of the minimax optimum. If $\xi = M_f(\check{\boldsymbol{\phi}})\,(M_f'(\check{\boldsymbol{\phi}}) = 0)$ then any *finite* value of p will yield the minimax solution!

This approach has been applied in the optimization of microwave networks [7], [19], [36], digital filters [37], and modeling problems [38], with values of p ranging between 1000 to 1 000 000.

B. Conditions for Optimality [12], [13]

Of great practical importance to engineers and applied mathematicians is the optimality of their approximation. Bandler and Charalambous [12], [13] derived the necessary and sufficient conditions for optimality in generalized nonlinear least pth optimization problems for $p \to \infty$. In the limit, the conditions for a minimax optimization are obtained [14]–[16].

In order for a point $\check{\boldsymbol{\phi}}$ to be a minimax optimum, it is necessary [and in the case of the convexity of $f_i(\boldsymbol{\phi})$ for $i \in I$, also sufficient], that

$$\sum_{i\in \hat{J}} u_i \nabla f_i(\check{\boldsymbol{\phi}}) = 0, \qquad u_i \geq 0 \tag{74}$$

$$\sum_{i\in\hat{J}} u_i = 1 \tag{75}$$

where

$$\hat{J} \triangleq \{i \mid f_i(\check{\boldsymbol{\phi}}) = M_f(\check{\boldsymbol{\phi}}), \qquad i \in I\}. \tag{76}$$

The multipliers u_i for $i \in \hat{J}$ are given by

$$u_i = \lim_{p\to\infty} \left(\frac{[f_i(\check{\boldsymbol{\phi}}_p)/M_f(\check{\boldsymbol{\phi}}_p)]^q}{\sum_{i\in\hat{J}} [f_i(\check{\boldsymbol{\phi}}_p)/M_f(\check{\boldsymbol{\phi}}_p)]^q}\right) \tag{77}$$

where $\check{\boldsymbol{\phi}}_p$ denotes the optimum parameter vector for particular values of $p\,(\check{\boldsymbol{\phi}}_\infty = \check{\boldsymbol{\phi}})$ and q as given by (70).

C. Charalambous–Bandler Algorithms [17], [18]

Based on the above ideas, Charalambous and Bandler were able to construct two new algorithms for minimax optimization [17], [18]. Unlike their original approach in which a very large value of p is required with the new algorithms, any finite value of p in the range $1 < p < \infty$

will produce extremely accurate minimax solutions. The computational procedure for both algorithms is as follows:

1) Assume the starting point ϕ^0 is given; set $\xi^1 = \min[0, M_f(\phi^0) + \epsilon]$, $r = 1$. ϵ is a small positive number.

2) Minimize with respect to ϕ the function

$$U(\phi,\xi^r) = M(\phi,\xi^r)\left(\sum_{i\epsilon K}\left(\frac{f_i(\phi) - \xi^r}{M(\phi,\xi^r)}\right)^q\right)^{1/q}, \quad \text{for } M(\phi,\xi^r) \neq 0 \tag{78}$$

$$= 0, \quad \text{for } M(\phi,\xi^r) = 0$$

where

$$M(\phi,\xi^r) \triangleq \max_{i\epsilon I}(f_i(\phi) - \xi^r) = M_f(\phi) - \xi^r \tag{79}$$

and

$$K \triangleq \begin{cases} J(\phi,\xi^r) = \{i \mid f_i(\phi) - \xi^r \geq 0, \quad i \in I\}, \\ \quad q = p, \text{ if } M > 0 \\ I, \; q = -p, \quad \text{if } M < 0. \end{cases} \tag{80}$$

3) Let $\check{\phi}^r$ denote the optimum parameter vector at the rth step. If Algorithm 1, go to 4; otherwise, if $M(\check{\phi}^r,\xi^r) < 0$, go to 4; or else set

$$\xi^{r+1} = \xi^r + \lambda^r M(\check{\phi}^r,\xi^r) = (1 - \lambda^r)\xi^r + \lambda^r M_f(\check{\phi}^r) \tag{81}$$

where

$$0 < \lambda^r < 1 \tag{82}$$

and go to 5.

4) Set

$$\xi^{r+1} = M_f(\check{\phi}^r) + \epsilon. \tag{83}$$

5) Convergence criterion: If $|\xi^{r+1} - \xi^r| < \eta$ stop; or else set $r = r + 1$ and go to 2. η is a small prescribed positive number.

Obviously, the larger the value of p is, the fewer will be the number of sequential optimizations required to reach a desired accuracy, but this does not mean that the number of function evaluations will be fewer. Algorithm 2 is different from Algorithm 1 as long as $M_f(\check{\phi}) > 0$, otherwise it is the same. The main difference is that in Algorithm 1 we try to push the maximum away from the level ξ^r at the rth iteration, while in Algorithm 2 we try to predict the value of $M_f(\check{\phi})$ by increasing ξ^r from zero appropriately. Due to this fact, $q = -p$ for $r = 2,3,\cdots$, for Algorithm 1, and for Algorithm 2, $q = p$ if $\xi^r < M_f(\check{\phi})$, and $q = -p$ if $\xi^r > M_f(\check{\phi})$. The reason for inclusion of ϵ is to avoid having $M = 0$, because in this case, when two or more of the f_i for $i \in I$ are equal the objective function's first derivatives are discontinuous.

It can be shown [17] that for both algorithms $|U(\check{\phi}^r,\xi^r)| \rightarrow 0$, $M_f(\check{\phi}^r) \rightarrow M_f(\check{\phi})$, as $r \rightarrow \infty$.

D. Error Function [7]

Define real error functions related to "upper" and "lower" specifications, respectively, as [7]

$$e_u(\phi,\psi) \triangleq w_u(\psi)(F(\phi,\psi) - S_u(\psi)) \tag{84}$$

$$e_l(\phi,\psi) \triangleq w_l(\psi)(F(\phi,\psi) - S_l(\psi)) \tag{85}$$

where

- $F(\phi,\psi)$ the approximating function (actual response);
- $S_u(\psi)$ an upper specified function (desired response bound);
- $S_l(\psi)$ a lower specified function (desired response bound);
- $w_u(\psi)$ an upper positive weighting function;
- $w_l(\psi)$ a lower positive weighting function.

In filter design problems, for example, $F(\phi,\psi)$ will be the response, ϕ may represent the network parameters, ψ could be frequency, $S_u(\psi)$ would refer to the passband specification, and $S_l(\psi)$ to the stopband specification. If $S_u(\psi) = S_l(\psi)$, $w_u(\psi) = w_l(\psi)$, then $e(\phi,\psi) = e_u(\phi,\psi) = e_l(\phi,\psi)$, which is the conventional case (single specified function).

In practice, we will evaluate all the functions at a finite discrete set of values of ψ taken from one or more closed intervals. Therefore, we will let, $e_{ui}(\phi) \triangleq e_u(\phi,\psi_i)$ for $i \in I_u$, and $e_{li}(\phi) \triangleq e_l(\phi,\psi_i)$ for $i \in I_l$, where

$$I_u \triangleq \{1,2,\cdots,n_u\}$$

$$I_l \triangleq \{n_u + 1, n_u + 2, \cdots, n_u + n_l\} \tag{86}$$

where n_u and n_l are the number of sample points over the upper specification and lower specification, respectively. Let

$$f_i(\phi) \triangleq \begin{cases} e_{ui}(\phi), & \text{for } i \in I_u \\ -e_{li}(\phi), & \text{for } i \in I_l \end{cases} \tag{87}$$

and

$$I \triangleq I_u \bigcup I_l. \tag{88}$$

If

$$M_f(\phi) \begin{cases} >0, & \text{the specification is violated} \\ =0, & \text{the specification is just met} \\ <0, & \text{the specification is satisfied.} \end{cases}$$

The objective function (69) has a very important property. If the objective function corresponding to the optimum solution with any finite value of p greater than one is positive (specification violated), then, at the optimum point for all other values of p (even with $p = \infty$, i.e., minimax), it will be positive. Similarly, if it is negative (specification satisfied), it will be negative for any other value of p. Thus, if we want to investigate whether a particular structure will satisfy design specifications in the minimax sense, any single least pth optimization will reveal this (even $p = 2$)!

Other interesting papers for minimax optimization are: Osborne and Watson [39], Bandler and Macdonald [40], Bandler *et al.* [41], Lasdon and Waren [42], and Klessig and Polak [43].

IV. Nonlinear Programming

A. The Nonlinear Programming Problem

The nonlinear programming problem can be stated as

$$\text{minimize } U(\boldsymbol{\phi}) \tag{89}$$

subject to

$$g_i(\boldsymbol{\phi}) \geq 0, \qquad i = 1,2,\cdots,m \tag{90}$$

where U is the generally nonlinear objective function of k parameters $\boldsymbol{\phi}$, and $g_1(\boldsymbol{\phi}),g_2(\boldsymbol{\phi}),\cdots,g_m(\boldsymbol{\phi})$ are, in general, nonlinear functions of the parameters. We will assume that all the functions are continuous with continuous partial derivatives, and that the inequality constraints $g_i(\boldsymbol{\phi}) \geq 0$, $i = 1,2,\cdots,m$ are such that a Kuhn–Tucker solution exists (see Lasdon [44] and Zangwill [45]).

Before the results of Section III were available, the nonlinear minimax optimization problems were solved by transforming them into nonlinear programs [46] and solved by well-established methods such as the barrier-function method of Fiacco and McCormick [47], [48]. Other methods of solving the resulting nonlinear programs include the repeated application of linear programming to suitably linearized versions of the nonlinear problem [39]. Very recently, Bandler and Charalambous transformed the nonlinear programming problem into an equivalent unconstrained minimax problem [21].

B. Bandler–Charalambous Algorithm [21]

Consider the problem of minimizing the unconstrained function

$$V(\boldsymbol{\phi},\boldsymbol{\alpha}) = \max_{1\leq i\leq m} [U(\boldsymbol{\phi}),U(\boldsymbol{\phi}) - \alpha_i g_i(\boldsymbol{\phi})] \tag{91}$$

where

$$\boldsymbol{\alpha} \triangleq [\alpha_1\alpha_2\cdots\alpha_m]^T \tag{92}$$

and

$$\alpha_i > 0, \qquad i = 1,2,\cdots,m. \tag{93}$$

Under the stated assumptions, the following can be proved [21].

If the Kuhn–Tucker necessary conditions for optimality of the nonlinear programming problem are satisfied at $\check{\boldsymbol{\phi}}$, then positive $\alpha_1,\alpha_2,\cdots,\alpha_m$ can be found satisfying

$$\sum_{i=1}^{m} \frac{u_i}{\alpha_i} < 1 \tag{94}$$

such that $\check{\boldsymbol{\phi}}$ satisfies the necessary conditions for optimality of $V(\boldsymbol{\phi},\boldsymbol{\alpha})$ with respect to $\boldsymbol{\phi}$, where $u_1,u_2,\cdots,u_m$ are the Kuhn–Tucker multipliers.

It is well known that $u_1,u_2,\cdots,u_m$ are specific nonnegative numbers, so that sufficiently large positive $\alpha_1,\alpha_2,\cdots,\alpha_m$ must be chosen to satisfy (94). Since $u_1,u_2,\cdots,u_m$ are not known in advance, one may not be able to forecast their values. If insufficiently large values of $\alpha_1,\alpha_2,\cdots,\alpha_m$ are chosen although a valid minimum of $V(\boldsymbol{\phi},\boldsymbol{\alpha})$ may exist, the constraints may not be satisfied at that point. A possible implementation is to use the results of Section III.

Let

$$f_i(\boldsymbol{\phi}) \triangleq U(\boldsymbol{\phi}) - \alpha_i g_i(\boldsymbol{\phi}), \qquad i = 1,2,\cdots,m \tag{95}$$

and

$$f_0(\boldsymbol{\phi}) = U(\boldsymbol{\phi}) \tag{96}$$

then the problem is to minimize

$$\max_{i\in I} f_i(\boldsymbol{\phi}) \tag{97}$$

where $I \triangleq \{0,1,\cdots,m\}$. Therefore, any of the three methods described in Section III can be used to solve the minimax problem for selected values of $\boldsymbol{\alpha}$. If the optimum parameter vector obtained is such that at least one of the constraints is violated, the elements of $\boldsymbol{\alpha}$ are increased, and the optimization procedure repeated. In practice, a tolerance for violated constraints should be specified.

There is no need to distinguish between feasible and nonfeasible regions. Due to this fact, equality constraints can be readily handled by transforming each one of them into two inequality constraints (e.g., if $h(\boldsymbol{\phi}) = 0$, we can transform it into the following two inequalities: $h(\boldsymbol{\phi}) \geq 0$, $-h(\boldsymbol{\phi}) \geq 0$).

C. Constrained Minimax Optimization I [17]

Let us suppose that we want to minimize $M_f(\boldsymbol{\phi})$ given by (61) subject to the constraints

$$g_i(\boldsymbol{\phi}) \geq 0, \qquad i = m+1,\cdots,m+n \tag{98}$$

It is well known that the nonlinear minimax optimization can be transformed into the nonlinear programming problem

$$\text{minimize } z \text{ (a new independent parameter)} \tag{99}$$

subject to

$$z - f_i(\boldsymbol{\phi}) \geq 0, \qquad i = 1,2,\cdots,m \tag{100}$$

and

$$g_i(\boldsymbol{\phi}) \geq 0, \qquad i = m+1,\cdots,m+n. \tag{101}$$

As we have already seen this nonlinear programming problem can be transformed into an equivalent unconstrained minimax problem, in which case the algorithm presented in Section III can be used.

D. Constrained Minimax Optimization II

Let us suppose that we have the same problem as in the previous subsection. Consider the minimization of the following unconstrained minimax function:

$$W(\boldsymbol{\phi},\boldsymbol{w}) = \max_{1\leq i\leq m+n} f_i(\boldsymbol{\phi}) \tag{102}$$

where

$$f_i(\boldsymbol{\phi}) \triangleq -w_i g_i(\boldsymbol{\phi}), \qquad i = m+1,\cdots,m+n \tag{103}$$

$$\boldsymbol{w} \triangleq [w_{m+1},\cdots,w_{m+n}] \tag{104}$$

$$w_i > 0, \qquad i = m+1,\cdots,m+n. \tag{105}$$

If at the constrained optimum $M_f(\check{\phi}) > 0$ and at least one of the constraints is active, the optimum point $\check{\phi}^w$ which the method will give will be nonfeasible, but $M_f(\check{\phi}^w) < M_f(\check{\phi})$. If $M_f(\check{\phi}) < 0$, then the optimum which the method will give will be feasible. If reasonably large values of the components of the vector w are used, the optimum point of this method will be quite close to true optimum. The resulting solution would be meaningful in the engineering sense since tradeoffs between response specifications and design constraints will be obtained.

Application of these two methods to microwave engineering problems can be found in [19] and [20]. Other interesting papers on nonlinear programming are given in [28], [30], and [47]–[58].

V. Example

E. Lumped-Distributed Active Filter

A third-order lumped-distributed active low-pass filter, in the form of the network shown in Fig. 1, is considered as an example. The problem is to be solved for minimax results in 3 ways:

1) an attenuation and ripple in the passband, [0,0.7] rad/s, of less than 1 dB, while the attenuation in the stopband, [1.415, ∞] rad/s, is at least 30 dB (second amplifier not included);
2) keeping the attenuation and the ripple in the passband at 1 dB and obtaining the best stopband response;
3) minimizing the attenuation and ripple in the passband subject to at least 30-dB attenuation in the stopband.

Problem 1 has been previously studied from a computer-aided design point of view, by Mokari-Bolhassan and Trick [59], and Problem 2 by Newcomb [60] by using synthesis methods. For both problems their results were not optimum in the minimax sense.

The node equations for the circuit are

$$\begin{bmatrix} y_{22} + j\omega C_1 & -(y_{22}+y_{12}) & 0 \\ -\left(y_{22}+y_{12}+\dfrac{A}{R_0}\right) & y_{11}+y_{22}+y_{12}+y_{21}+\dfrac{1}{R_0} & 0 \\ -\dfrac{A}{R_1} & 0 & \dfrac{1}{R_1}+j\omega C_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} -y_{12}V_S \\ (y_{11}+y_{12})V_S \\ 0 \end{bmatrix} \quad (106)$$

where y_{11}, y_{12}, y_{21}, and y_{22} are the y parameters of the uniform RC distributed line

$$V_0 = \frac{1}{A} V_3. \quad (107)$$

It was decided to use 9 sample points in the passband, in radians per second:

{0,0.1,0.2,0.3,0.4,0.5,0.6,0.65,0.7}

and 17 sample points in the stopband

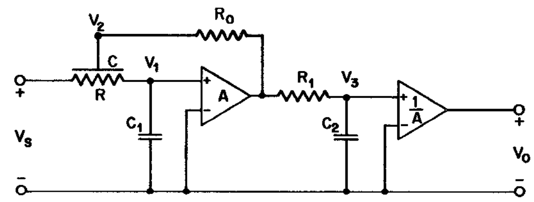

Fig. 1. Third-order lumped-distributed active lowpass filter.

{1.415,1.5,1.6,1.7,1.8,1.9,2.0,2.1,2.2,2.3,2.4,2.5,2.6,2.7, 2.8,2.9,3.0}.

For all three problems

$$e_{ui} = \begin{cases} w_{ui}(F_i - r), & i = 1,2,\cdots,9 \\ w_{ui}(F_i + 30), & i = 10,11,\cdots,26 \end{cases} \quad (108)$$

$$e_{li} = w_{li}(F_i + 1), \qquad i = 27,28,\cdots,35 \quad (109)$$

where F is the gain in decibels, and r a small number whose significance will become apparent later. From (106), it can be seen that C_2 and R_1 appear together as C_2R_1 and therefore represent one variable. Hence C_2 was kept fixed at the value 2.62 as given by Mokari-Bolhassan and Trick.

For Problem 1, all the weighting factors were set equal to 1, $r = 0$, and

$$\phi = [A \quad R \quad C \quad R_0 \quad R_1 \quad C_1]^T.$$

For Problem 2, the weighting over the passband was set equal to 1, and over the stopband equal to 10^{-4} and $r = 10^{-3}$. The reason for the small weight in the stopband is to give more emphasis to the "deviation" in the stopband. The small value of r is used because at zero frequency the function F is equal to zero and by using the approach described, would lead to $M = 0$ if $r = 0$. To avoid the possibility of the error at zero frequency from becoming active, it is necessary that the following inequality holds at the optimum point:

$$r > |M(\check{\phi},\xi)|. \quad (110)$$

For Problem 3 the weighting over the upper specification was set equal to 1, over the lower specification to 10^{-4} and $r = 10^{-3}$. The same comments as those in Problem 2 hold.

For Problems 2 and 3,

$$\phi = [A \quad C \quad R_1 \quad C_1]^T$$

with $R_0 = 1$ and $R = 17.786$.

Optimization using the Fletcher method [3], in accordance with the ideas discussed in Section III, with $p = 2$, 10, 100, 1000, 10 000 (p is increased after each optimum is

TABLE I
OPTIMIZATION OF THE CIRCUIT SHOWN IN FIG. 1

Parameters	A	R	C	R_0	R_1	C_1	C_2
Starting point (MBT[a])	1.142	17.786	0.427	1.0	1.0	0.067	2.62
Optimum point							
Problem 1	1.00339	21.75063	0.47719	0.61713	0.95378	0.04525	kept fixed
Problem 2	1.25659	kept fixed	0.39304	kept fixed	1.01968	0.08975	kept fixed
Problem 3	1.11832	kept fixed	0.39287	kept fixed	0.68946	0.06639	kept fixed

[a] Mokari-Bolhassan and Trick (1971).

reached) gave the results shown in Table I. In all the cases, the following parameter transformation was used to keep the parameters positive:

$$\phi_i = \exp \phi_i' \tag{111}$$

where ϕ_i' is used as a variable instead of ϕ_i. The adjoint network approach was used for the calculation of the gradients of F with respect to $\boldsymbol{\phi}'$ [61], and can be found in [23].

The starting point in all three problems was the one which Mokari-Bolhassan and Trick (MBT on the Figs. 2–4) called their optimum. The starting and the optimum responses are depicted in Figs. 2–4, from which it can be seen that the results of Mokari-Bolhassan and Trick are not optimum in the minimax sense. It can be easily checked that (110) is satisfied for Problems 2 and 3.

It is interesting to note that Problems 2 and 3 fall in the framework of nonlinear minimax approximation with nonlinear constraints. That is, Problem 2 can be phrased as "minimize the maximum of 17 nonlinear functions subject to 18 nonlinear constraints and 4 linear constraints" and Problem 3 as "minimize the maximum of 9 nonlinear functions subject to 26 nonlinear constraints and 4 linear constraints," which obviously is not so easy.

Problems 2 and 3 have also been solved by creating violated specifications. For Problem 2, we set $S_u = 0$ dB, $S_l = 1$ dB over the passband, $S_u < -35.3$ dB over the stopband, and use a reasonably large weighting factor over the passband. For Problem 3, we set $S_u = S_l = 0$ dB over the passband, $S_u = -30$ dB over the stopband, and we use a reasonably large weighting factor over the upper specification. For both problems the zero frequency can be neglected. The results obtained were in very good agreement with those shown in Table I.

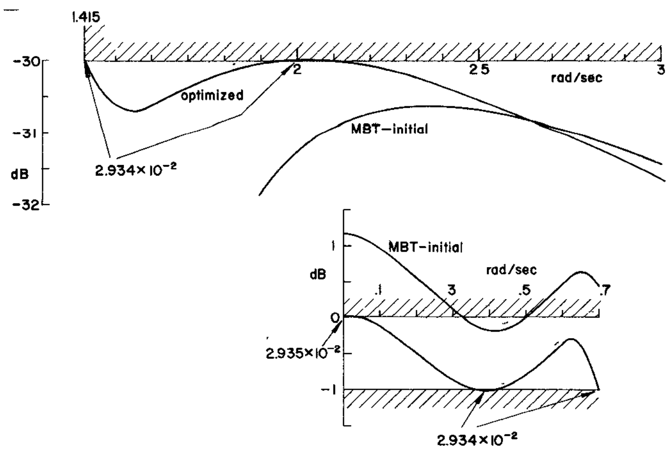

Fig. 2. Optimized gain of the circuit of Fig. 1 subject to the constraints imposed for Problem 1.

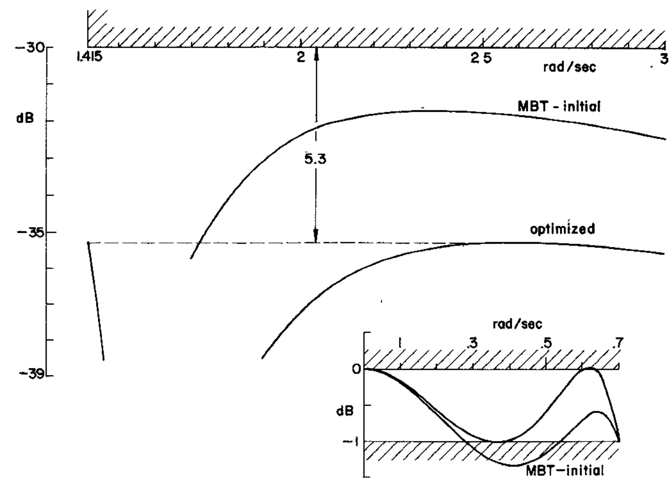

Fig. 3. Optimized gain of the circuit of Fig. 1 subject to the constraints imposed for Problem 2.

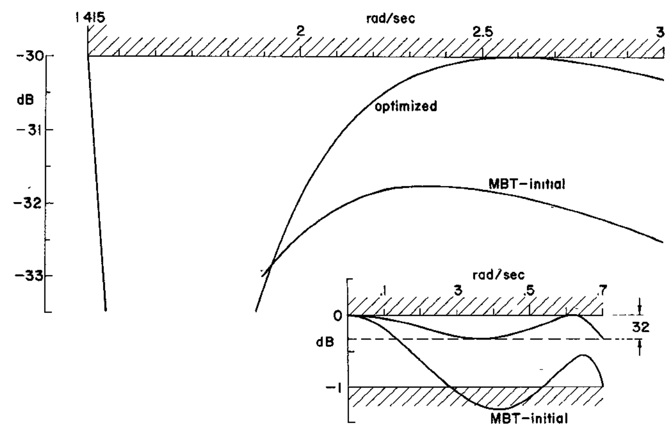

Fig. 4. Optimized gain of the circuit of Fig. 1 subject to the constraints imposed for Problem 3.

VI. CONCLUSIONS

When preparing this paper, the author had in mind the question of future investigation. Four separate subjects are recommended for future research.

Extrapolation techniques can be applied for both algorithms presented in Section III on ϕ^r and ξ^r separately or simultaneously, which can accelerate the convergence. This will also have an impact on the new method of nonlinear programming. Fiacco and McCormick [53], for example, have applied extrapolation techniques in their method of solving the nonlinear programming problem.

One of the main disadvantages of the original Fiacco–McCormick approach to nonlinear programming is that the function is undefined outside the feasible region. What this author proposes is a "negative to positive barrier method" through the constraint boundary. In other words, the function is to be finite and positive outside the feasible region and negative inside the feasible region, if the starting point is feasible.

An automated method is needed to recognize a redundancy in the variables of a design problem. A redundant variable can cause ill-conditioning. An excellent example is the lumped-distributed active filter studied in Section V. The parameters C_2 and R_1 appear together as C_2R_1 and therefore represent one variable. Fixing C_2 at 2.62, the Fletcher method [3] took less than 1 min of CDC 6400 computer time to reach the optimum with the value of p equal to 10 000. The main numerical difficulty with such examples is that the Hessian matrix of the objective function becomes singular at those points where the first derivative of the function with respect to the combined variable is zero (note that this might happen at points which are not optimal). There is an infinite number of points where this can occur. Since most of the algorithms used for optimization assume that the Hessian matrix is nonsingular, they will be very slow or they might even stall. Another interesting problem of a redundant parameter is given by Markettos [62]. Following the ideas developed in Section III and making an error analysis for large p, a way to reduce the computational effort might be found.

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Cascaded Network Optimization Program

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***Abstract*—A user-oriented computer program package is presented that will analyze and optimize certain cascaded linear time-invariant electrical networks such as microwave filters and all-pass networks. The program is organized in such a way that future additions or deletions of performance specifications, constraints, optimization methods, and circuit elements are readily implemented. Presently, a variety of two-port lumped and distributed elements, all-pass *C*-type sections and all-pass *D*-type sections can be treated as fixed or variable between upper and lower bounds on the parameters. Adjoint network sensitivity formulas are incorporated. The Fletcher–Powell or Fletcher optimization methods can be called to optimize in the least *p*th sense of Bandler and Charalambous an objective function incorporating simultaneously, at the user's discretion, input reflection coefficient, insertion loss, group delay, and the parameter constraints (if any). The program is particularly flexible in the way in which response specifications are handled at any number of, in general, overlapping frequency bands. The package, which is written in Fortran IV, has been tested on a CDC 6400 digital computer.**

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I. Introduction

A USER-ORIENTED computer program package is presented that will analyze and optimize certain cascaded linear time-invariant networks such as microwave filters and all-pass networks in the frequency domain.